Exponential Type Complex and non-Hermitian Potentials within Quantum Hamilton-Jacobi Formalism

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Abstract

PT-/non-PT-symmetric and non-Hermitian deformed Morse and Pöschl-Teller potentials are studied first time by quantum Hamilton-Jacobi approach. Energy eigenvalues and eigenfunctions are obtained by solving quantum Hamilton-Jacobi equation.

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1 Introduction

Recently, the so called PT-symmetric quantum mechanics has attracted wide attention [1]. In Bender and Boettcher's work in 1998 [1], it was shown that the class of non-Hermitian, Hamilton operators such as $H = p^2 + x^2(ix)^\epsilon (\epsilon > 0)$ has a real spectrum due to its PT-symmetry where P and T are the parity and time reversal operators respectively [2]. Exact solution of the Schrödinger equation for various potentials which are complex are generally of interest. It is also known that PT-symmetry does not necessarily lead to completely real spectrum, and an extensive kind of potentials of real or complex form are being faced with in various fields of physics. In particular, the spectrum of the Hamiltonian is real if PT-symmetry is not spontaneously broken. Recently, Mostafazadeh has generalized PT symmetry by pseudo-Hermiticity [3]. In fact, a Hamiltonian of this type is said to be η- pseudo Hermitian if $H^+ = \eta H \eta^{-1}$, where + denotes the operator of adjoint. In [4] new class of non-Hermitian Hamiltonians with real spectra was proposed which are obtained using pseudo-symmetry. Moreover, completeness and orthonormality conditions for eigenstates of such potentials are proposed [5]. In the study of PT-invariant potentials various techniques from a great variety of quantum mechanical fields have been applied such as variational methods, numerical approaches, Fourier analysis, semi-classical estimates, quantum field theory and Lie group theoretical approaches [5-14]. In addition, PT-symmetric and non-PT symmetric and also non-Hermitian potential cases such as oscillator type potentials [15], a variety of potentials within the framework of SUSYQM [16-19], exponential type screened potentials [20], quasi/conditionally exactly solvable ones [21], PT-symmetric and non-PT symmetric and also non-Hermitian potential cases within the framework of SUSYQM via Hamiltonian Hierarchy Method [22] and some others are studied [23-25].

The QHJ formalism, which is a formulation of quantum mechanics was investigated as a theory related to the classical transformation theory [26-27]. It was formulated by Leacock and Padgett in 1983 [28-29]. Within the Quantum Hamiltonian Jacobi approach (QHJ), which follows classical mechanics, not only the the energy spectrum of exactly solvable (ES) and quasi-exactly solvable (QES) models in quantum mechanics but eigenfunctions can also be determined [30-36]. The advantage of this method is that it is possible to determine the energy eigenvalues without having to solve for the eigenfunctions. In this formalism, singularity structure of the quantum momentum function

$p(x)$ which is a quantum analog of classical momentum function p_c determines the eigenvalues of the Hamiltonian. An exact quantization condition is formulated as a contour integral, representing the quantum action variable, in the complex plane. The quantization condition leads to the number nodes of the wave function. The wavefunction is related to the quantum momentum function (QMF). The equation satisfied by the QMF is a non-linear differential equation, called as quantum Hamilton-Jacobi equation. There is a boundary condition in the limit QMF which is used to determine physically acceptable solutions for the QMF [29-37]. In the applications, Ranjani and her collaborators applied the QHJ formalism, to Hamiltonians with Khare-Mandal potential and Scarf potential, characterized by discrete parity and time reversal (PT) symmetries [31].

The purpose of the present work has been to apply the QHJ formalism to the Hamiltonian in one dimension with non-hermitian exponential type potentials in order to see possible singularities for the QMF which determine eigenvalues and convenient eigenfunctions.

The organization of the paper is as follows. In Sec. II, we briefly introduce the Quantum Hamilton-Jacobi formalism. In Sec. III and IV,solutions of PT-/non-PT-symmetric and non-Hermitian forms of the well-known potentials are presented by using QHJ method. We discuss the results in Sec. V.

2 Quantum Hamilton-Jacobi Formalism

In quantum theory, one assumes that function $W(x, E)$ satisfies $(2m = 1)$ [33],

$$- i\hbar \frac{\partial^2 W(x, E)}{\partial x^2} + \left[\frac{\partial W(x, E)}{\partial x} \right]^2 = (E - V(x)) \tag{1}$$

Eq. (1) will be called as the QHJ equation. The momentum function

$$p(x, E) = \frac{\partial W(x, E)}{\partial x} \tag{2}$$

will be called as the QMF. In the limit $\hbar \rightarrow 0$, the QHJ equation turns into the classical Hamilton-Jacobi equation. Then, QMF turns into the classical momentum function in the $\hbar \rightarrow 0$ limit:

$$p(x, E) \rightarrow p_c(x, E) = \sqrt{E - V(x)} \tag{3}$$

In terms of $p(x, E)$ the QHJ equation, Eq.(1) can be written as

$$p^2(x, E) - i\hbar p'(x, E) - [E - V(x)] = 0. \tag{4}$$

Leacock and Padgett [28,29] proposed using the following quantization condition for the bound states in order to obtain eigenvalues. C is a contour that encloses the moving poles between the classical turning points and the integral

$$J(E) = \frac{1}{2\pi} \oint_C p(x) \, dx \tag{5}$$

is called the quantum action variable. More details can be found in the paper of Bhalla et all [30-37]. Then

$$J = n\hbar = J(E) \tag{6}$$

gives the exact energy eigenvalues ($n = 0, 1, 2, ...$) [28-33]. Leacock [28,29] defines the wave function in order to connect QHJ equation to the Schödinger equation,

$$\psi(x, E) \equiv exp\left[\frac{i}{\hbar}W(x, E)\right] \tag{7}$$

hence $\psi(x, E)$ satisfies the Schödinger equation and the physical boundary conditions. The quantization condition becomes [33]

$$\oint_C p(x, E)dx = 2\pi i \sum_k (Res)_k = nh \tag{8}$$

where $\sum_k (Res)_k$ is sum of the residues. In the QHJ equation, if $V(x)$ has a singular point, $p(x, E)$ will also have singular point in that zone [33]. These singularities are known as fixed singular points which are energy independent. Other types of singular points are the moving singular points. They can only be poles with residue $-i\hbar$. Suppose $b \neq 0$ then, moving singularities are in the form of [30-37]

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$$p(x, E) \sim \frac{b}{(x - x_0)^r} + \ldots \tag{9}$$

in the QHJ equation. If the potential is not singular at $x = x_0$ then r must be equal to one and $b = -i\hbar$ [33].

3 Generalized Morse Potential

The generalized Morse potential is given by [19]

$$V(x) = V_1 e^{-2\alpha x} - V_2 e^{-\alpha x} \tag{10}$$

In order to apply QHJ method, we write the potential relation in Eq.(4) ($\hbar = 2m = 1$)

$$p^2 - ip' - [E - V_1 e^{-2\alpha x} + V_2 e^{-\alpha x}] = 0 \tag{11}$$

Substitution of the transformation of $y = \sqrt{V_1} e^{-\alpha x}$ in Eq.(11) gives:

$$p^2(y, E) + i\alpha y p'(y, E) - \left[E - y^2 + \frac{V_2}{\sqrt{V_1}} y \right] = 0 \tag{12}$$

Define $p = i\alpha y \phi$ and $\chi = \phi + \frac{1}{2y}$ in order to transform Eq.(11) into a Riccati type differential equation as,

$$\chi' + \chi^2 + \frac{1}{4y^2} + \frac{1}{\alpha^2 y^2} \left[E - y^2 + \frac{V_2}{\sqrt{V_1}} y \right] = 0 \tag{13}$$

As it can be seen from Eq. (13), χ has a pole only at $y = 0$ and for $y = 0$ define χ as,

$$\chi = \frac{b_1}{y} + a_0 + a_1 y \tag{14}$$

Substitute Eq.(14) in (13) and equate coefficients of $\frac{1}{y^2}$ yields

$$b_1 = \frac{1}{2\alpha}(\alpha \pm 2\sqrt{-E}) \tag{15}$$

When it comes to the discussion of the behaviour of χ at infinity, one expands χ as:

$$\chi = a_0 + \frac{\lambda}{y} + \frac{\lambda_1}{y^2} \tag{16}$$

and find λ as

$$\lambda = \pm \frac{V_2}{2\alpha\sqrt{V_1}} \tag{17}$$

One can see that the behaviour of χ is $\frac{b_1+n}{y}$ for large y. Hence,

$$b_1 + n = \lambda \tag{18}$$

In order to find the wavefunction, $\chi(y)$ can be written as the sum of the Laurent expansions around different singular points, plus a constant C_1. Hence

$$\chi(y) = \frac{b_1}{y} + \frac{P_n'(y)}{P_n(y)} + C_1 \tag{19}$$

where $P_n(y)$ is a n th degree polynomial. Substitute Eq. (19) in (13) and get

$$\frac{P_n''}{P_n} + \frac{2P_n'}{P_n}\left(\frac{2b_1}{y} + 2C_1\right) + \left(\frac{b_1^2 - b_1 + E/\alpha^2 + 1/4}{y^2} + \frac{2b_1 C_1}{y} + C_1^2 - \frac{1}{\alpha^2} + \frac{V_2}{\alpha^2 y\sqrt{V_1}}\right) = 0. \tag{20}$$

For large y one can find $C_1 = \pm\frac{1}{\alpha}$. The wave function in terms of χ can be written by using eq. (19) and (7) as

$$\psi(y) = exp\left(\int \left(\frac{b_1}{y} + \frac{P_n'}{P_n} - \frac{1}{\alpha} - \frac{1}{2y}\right)dy\right). \tag{21}$$

In Eq. (21) the correct value of C_1 is used as $C_1 = -\frac{1}{\alpha}$ because of the condition for the wavefunction which is known as $y \to \infty$, $\psi(y) \to 0$. It is seen from Eq. (15) that b_1 has two values and no particular value has been chosen. Using Eq. (18), the energy eigenvalues for any n-th state become,

$$E_n = -\frac{\alpha^2}{4}\left[-(2n+1) + \frac{V_2}{\alpha\sqrt{V_1}}\right]^2 \tag{22}$$

If we use Eq.(22),(15) and $C_1 = -1/\alpha$ in Eq.(20), it becomes

$$yP_n''(y) + (\frac{V_2}{2\alpha\sqrt{V_1}} + 1 - y)P_n'(y) + nP_n(y) = 0 \tag{23}$$

which is a Laguerre differential equation. Therefore, the wavefunction is obtained as

$$\psi_n(y) = C_n \, e^{-\frac{y}{\alpha}} \, y^{(-n\pm\frac{V_2}{2\alpha\sqrt{V_1}})} \, L_n^{\frac{V_2}{2\alpha\sqrt{V_1}}}(y). \tag{24}$$

where C_n is a normalization constant and $L_n^{\frac{V_2}{2\alpha\sqrt{V_1}}}(y)$ are Laguerre functions. The wavefunction satisfies the boundary condition that is $y \to \infty$, $\psi(y) \to 0$.

3.1 Non-PT symmetric and non-Hermitian Morse Potential

In equation (10), if the potential parameters are defined as $V_1 = (A + iB)^2$, $V_2 = (2C + 1)(A + iB)$ and $\alpha = 1$, then the potential becomes [19],

$$V(x) = (A + iB)^2 e^{-2x} - (2C + 1)(A + iB)e^{-x} \tag{25}$$

where A, B and C are arbitrary real parameters and $i = \sqrt{-1}$. The QHJ equation is

$$p^2 - ip' - [E - (A + iB)^2 e^{-2x} + (2C + 1)(A + iB)e^{-x}] = 0 \tag{26}$$

Using the transformation in the form of $y = (A + iB)e^{-x}$ in the Eq. (26), then using $p(y) = iy\phi$ and $\chi = \phi + \frac{1}{2y}$, Eq. (26) becomes

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$$\chi' + \chi^2 + \frac{1}{4y^2} + \frac{1}{y^2}\left[E - y^2 + (2C+1)y\right] = 0 \tag{27}$$

As it is seen from Eq. (27), χ has a pole only at $y = 0$ and for $y = 0$ define χ for the Eq.(27) as,

$$\chi = \frac{b_1}{y} + a_0 + a_1 y \tag{28}$$

Using the Eq. (28) in (27), b_1 is found as

$$b_1 = \frac{1}{2} \pm i\sqrt{\frac{|4E|}{2}} \tag{29}$$

Following the same procedure that is given in section 3, energy is found as

$$E_n = -(n - C)^2. \tag{30}$$

Now choose the potential parameters in Eq.(10) as V_1 is real and $V_2 = A + iB$, the Morse potential can be written in the following form

$$V(x) = V_1 e^{-2i\alpha x} - (A + iB)e^{-i\alpha x} \tag{31}$$

and following the same procedure, the energy is obtained as

$$E_n = \alpha^4 \left[(n + 1/2) - \frac{A + iB}{2\alpha\sqrt{|-V_1|}}\right]^2 \tag{32}$$

According to Eq. (32), the spectrum is real in the case of $Im(V_2) = 0$.

3.2   PT symmetric and non-Hermitian Morse Potential

When $\alpha = i\alpha$ and V_1, V_2 are real, the Morse potential becomes

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$$V(x) = V_1 e^{-2i\alpha x} - V_2 e^{-i\alpha x} \tag{33}$$

Thus, the energy eigenvalues are obtained as

$$E_n = \alpha^4 \left[(n + \frac{1}{2}) + \frac{V_2}{2\alpha\sqrt{|-V_1|}} \right]^2 \tag{34}$$

If we take the parameters of Eq.(39) as $V_1 = -\omega^2$, $V_2 = D$ and $\alpha = 2$ then, corresponding eigenvalues for any n-th state are obtained as

$$E_n = (2n + 1 + \frac{D}{2\omega})^2 \tag{35}$$

are consistent with the results [10-11,19].

4 Pöschl-Teller Potential

The general form of the Pöschl-Teller potential is

$$V(x) = -4V_0 \frac{e^{-2\alpha x}}{(1 + qe^{-2\alpha x})^2} \tag{36}$$

The QHJ equation is given as

$$p^2 - ip' - \left(E + 4V_0 \frac{e^{-2\alpha x}}{(1 + qe^{-2\alpha x})^2} \right) = 0 \tag{37}$$

If we take $y = \pm i\sqrt{q}e^{-\alpha x}$ and use the transformations as $p = -i\alpha y\phi$ and $\chi = \phi + \frac{1}{2y}$, the QHJ equation turns into

$$\chi' + \chi^2 + \frac{1}{4y^2} + \frac{1}{\alpha^2 y^2} \left[E - \frac{4V_0}{q} \frac{y^2}{(1 - y^2)^2} \right] = 0. \tag{38}$$

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As it is seen from Eq.(38), χ has poles at $y = 0$ and ± 1. χ is expanded at $y = 0$ and b_1 is found

$$b_1 = \frac{1}{2\alpha}(\alpha \pm 2\sqrt{-E}) \tag{39}$$

At $y = 1$, one can expand χ as

$$\chi = \frac{b_1'}{1 - y} + a_0' + a_1'(1 - y) \tag{40}$$

and b_1' is found as

$$b_1' = \frac{1}{2q\alpha}(q\alpha \pm \sqrt{\alpha^2 q^2 + 8qV_0}) \tag{41}$$

At $y = -1$, one can expand χ as

$$\chi = \frac{b_1''}{1 + y} + a_0'' + a_1''(1 + y) \tag{42}$$

Substitute Eq. (42) in (38), to obtain $b_1' = b_1''$. One can look at the behavior of χ at infinity with expanding χ as

$$\chi = A_0 + \frac{\lambda}{y} + \frac{\lambda}{y^2} \tag{43}$$

From Eq. (43) and (38), λ is found as

$$\lambda = \frac{1}{2\alpha}(\alpha \pm 2\sqrt{-E}) \tag{44}$$

and behavior of χ is $\frac{b1 + b_1' + b_1'' + 2n}{y}$ for large y. Hence,

$$\lambda = b_1 + b_1' + b_1'' + 2n \tag{45}$$

In order to find the wavefunctions, χ is written as

$$\chi = \frac{b_1}{y} + \frac{b_1'}{1-y} + \frac{b_1''}{1+y} + \frac{P_n'(y)}{P_n(y)} + C_2 \tag{46}$$

Substituting Eq. (46) in (38), C_2 can be found as $C_2 = 0$ for large y. The wavefunction can be written as

$$\psi = exp\left(\int \left(\frac{2b_1 - 1}{2y} + \frac{b_1'}{1-y} + \frac{b_1''}{1+y} + \frac{P_n'(y)}{P_n(y)}\right) dy\right) \tag{47}$$

If we look at Eq. (47), b_1 and b_1' have two values and both of them gives appropriate results for energy spectrum and wavefunction, no particular value has been chosen. Thus, residues are given as $b_1 = \frac{1}{2\alpha}(\alpha \pm 2\sqrt{-E})$ and $b_1' = b_1'' = \frac{1}{2q\alpha}(q\alpha \pm \sqrt{\alpha^2 q^2 + 8qV_0})$. Finally the energy is obtained by using Eq. (45) as

$$E_n = -\frac{\alpha^2}{4}\left((2n+1) \pm \sqrt{1 + \frac{8V_0}{q\alpha^2}}\right)^2 \tag{48}$$

Using Eqs. (47,48) and (38), one can find the wave function as

$$\psi_n(y) = N \ y^{-(n-1/2)\pm\gamma}(1-y^2)^{\frac{1}{2}(1\pm\gamma)} \ P_n^{-\nu_2-\frac{1}{2}, \ \nu_2-\frac{1}{2}}(y) \tag{49}$$

where N is a normalization constant, $\gamma = \sqrt{1 + \frac{8V_0}{q\alpha^2}}$, $P_n^{-\nu_2-\frac{1}{2},\nu_2-\frac{1}{2}}(y)$ stands for Jacobi polynomials and $\nu_2 = \sqrt{\frac{8V_0}{q\alpha^2}}$. If we look at Eq.(49), there are three cases for physical solutions because of the wavefunction that satisfies the boundary condition as $y \to \infty$, $\psi(y) \to 0$. If $-(n-1/2) \pm \gamma < 0$ and $1 \pm \gamma > 0$, it should be $1 \pm \gamma > |-(n-1/2) \pm \gamma|$. If $-(n-1/2) \pm \gamma < 0$ and $1 \pm \gamma < 0$, there is no restriction for the parameters and there are physical solutions in this case. The last case can be defined as; if $-(n-1/2) \pm \gamma > 0$, $1 \pm \gamma > |-(n-1/2) \pm \gamma|$ for appropriate solutions.

4.1 Non-PT symmetric and non-Hermitian Pöschl-Teller cases

Here, V_0 and q are complex parameters $V_0 = V_{0R} + iV_{0I}$ and $q = q_R + iq_I$ but α is a real parameter. Although the potential is complex and the corresponding Hamiltonian is non-Hermitian and also non-

PT symmetric, there may be real spectra if and only if $V_{0I}q_R = V_{0R}q_I$. When both parameters V_0 and q are taken pure imaginary, the potential turns out to be,

$$V(x) = -4V_0 \frac{2qe^{-4\alpha x} + i(1 - q^2 e^{-4\alpha x})}{(1 + q^2 e^{-4\alpha x})^2} \tag{50}$$

For simplicity, we use the notation V_0 and q instead of V_{0I} and q_I. In this case, we get the same energy eigenvalues as in Eq.(48). If q is an arbitrary real parameter and $V_0 \Rightarrow iV_0$ also $\alpha \Rightarrow i\alpha$ completely imaginary, the potential becomes

$$V(x) = -4V_0 \frac{(1 - q^2)sin2\alpha x + i(2q + (1 + q^2)cos2\alpha x)}{(1 + q^2)^2 + 4qcos2\alpha x(1 + qcos2\alpha x + q^2)} \tag{51}$$

and the corresponding energy eigenvalues become

$$E = \frac{\alpha^2}{4} \left[2n + 1 + \sqrt{1 + \frac{8V_0}{\alpha^2}} \right]^2 \tag{52}$$

For simplicity, let us take all three parameters α, q, V_0 purely imaginary. Then the potential takes the form

$$V(x) = -4V_0 \frac{(1 + q^2)sin2\alpha x + 2q + i((1 - q^2)cos2\alpha x)}{(1 + q^2)^2 + 4q^2(1 - cos^2 2\alpha x) + 4q(1 + q^2)sin2\alpha x} \tag{53}$$

and the energy becomes

$$E_n = \frac{\alpha^2}{4} \left[2n + 1 + \frac{1}{2\alpha q}\sqrt{\alpha^2 q^2 + (1 + q^2)V_0} \right]^2 \tag{54}$$

4.2   PT symmetric and non-Hermitian Pöschl-Teller cases

We choose parameters V_0 and q real and also $\alpha = i\alpha$. Then, the potential turns into

$$V(x) = -4V_0 \frac{(1+q^2)cos2\alpha x + 2q + i(q^2-1)sin2\alpha x}{(1+q^2)^2 + 4qcos2\alpha x((1+qcos2\alpha x + q^2)} \tag{55}$$

and corresponding energy spectrum is

$$E = -\frac{\alpha^2}{4} \left[2n + 1 + \sqrt{1 + \frac{8V_0}{\alpha^2}} \right]^2 \tag{56}$$

5 Conclusions

We have applied the PT-symmetric formulation to solve the Quantum Hamilton-Jacobi equation for Morse and Pöschl-Teller potentials in both real and complex forms. We have obtained the energy eigenvalues and the corresponding wave functions for different forms of these potentials within Quantum Hamilton-Jacobi formalism. The real energy spectra of the PT-/non-PT- symmetric complex valued non-Hermitian potentials have been obtained in case the potential parameters are restricted. It is also shown that the QHJ formalism is a good approach to obtain eigenfunctions and energy eigenvalues for a class of exponential type potentials discussed here within PT symmetric frame. As a result, we have pointed out that our exact results of complexified general Morse and Pöschl-Teller potentials may increase the number of applications of complex Hamiltonians with real energies in the extensive study of different quantum systems within the flexible Quantum Hamilton-Jacobi approach. Finally we should state that this work is the first application on the study of PT-symmetry for the Quantum Hamilton-Jacobi approach.

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